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Polarization entanglement purification using the temporal degree of freedom

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A B S T R A C T

Polarization entangled photon pairs are easily perturbed in noisy channels. We propose a polarization entanglement purification method using temporal degree of freedom, followed by the conventional iterative purification. The entanglement fidelity can be improved to any degree, and the steps needed are less than those using conventional iteration method.

Entangled photon pairs show non-local correlation properties, and they are important sources for quantum information processing tasks such as quantum cryptography [1], quantum teleportation [2], quantum dense coding [3], quantum error correction [4]. Polarization entangled photon pairs (PEPPs) are frequently used in quantum information due to their easy generation and manipulation. Whereas, the quality of the polarization entanglement is easily affected by the noises in the transmission channels, hence entanglement purification is necessary to produce high quality entanglement. Many purification schemes have already been proposed [5,6], and high fidelity can be recovered by iterative purification [7]. However, the efficiency, which is defined as the number ratio of the output pairs to the input pairs, is intrinsically low, because at least half of the particles are used as targets in every iteration step.

Recently, entanglement with multiple degrees of freedom (DOF) has been proposed for quantum communication without alignment in Ref. [8]. In this Letter, we propose a high efficiency scheme for polarization entanglement purification, by introducing entanglement on the temporal DOF to the PEPP. Bit flipped noises can be wiped off easily by choosing the right time window. Followed by conventional iterative purification, the fidelity can be improved to any degree. The total efficiency is high, and less iteration steps are needed.

The four Bell-states for PEPP are

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\big(|H\rangle_A|H\rangle_B \pm |V\rangle_A|V\rangle_B\big),$$
$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\big(|H\rangle_A|V\rangle_B \pm |V\rangle_A|H\rangle_B\big), \tag{1}$$

where H and V stand for the horizontal and vertical polarization, respectively. The subscript A and B denote the photons on Alice and Bob sides. If the initial PEPPs in pure state $|\Phi^+\rangle$ are transmitted through noisy channels, polarization perturbation changes the desired state into the mixture of the four in (1). In our scheme, one easily adds time delay t by insetting a birefringence material in each side of the channels, after the particles in the pure $|\Phi^+\rangle$ state have been generated. For example, the $H(V)$ polarized photon is along the slow (fast) axis of the material. Unavoidably, a phase delay ϕ is also added to the H photons. The $|\Phi^+\rangle$ state is changed into

$$|\Phi_t^+\rangle = \frac{1}{\sqrt{2}}\big(e^{i2\phi}|H,t\rangle_A|H,t\rangle_B + |V\rangle_A|V\rangle_B\big), \tag{2}$$

which shows entanglement on two DOFs—one is the polarization, the other is time-bin. To distinguish it from the PEPP, we add a subscript t to the state with entanglement on the temporal DOF. Since polarization and time-bin are in two direct product spaces, there exist some operations on time which are irrelevant to polarization states of photons, and vice versa. Unlike the PEPP, when the pairs are transmitted, perturbations transform the initial pure state in (2) into the mixture composed of the following eight states:

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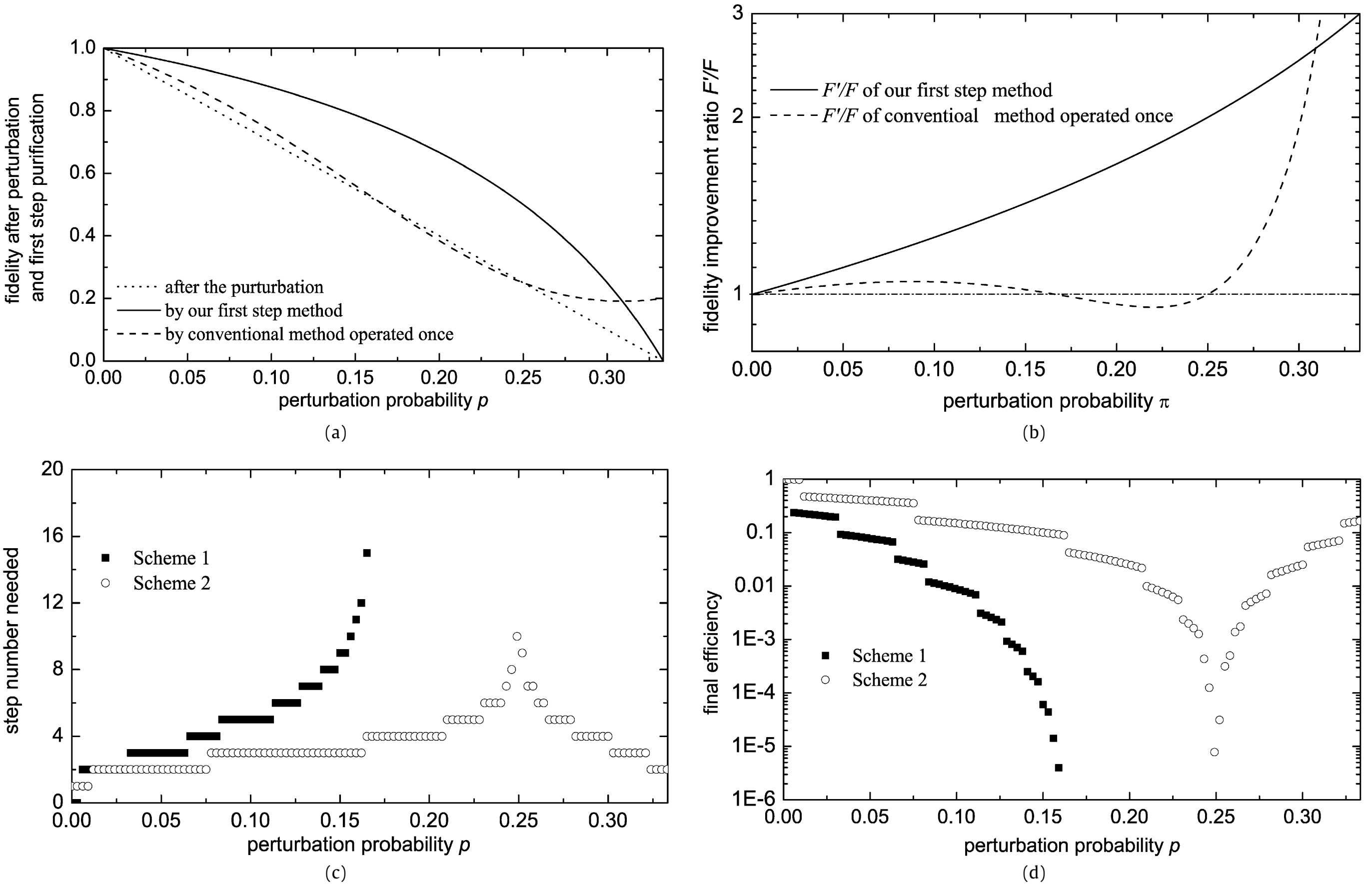

(a) (b)

(c) (d)

Fig. 1. The comparison between two schemes when the entangled photon pairs are exposed to single side perturbation. Scheme 1: Transmission using $|\Phi^+\rangle$ state and distillation using conventional iterative purification; Scheme 2: Transmission and distillation using our method, followed by conventional iterative purifications. (a) The fidelities F after the states been perturbed, which are the same for PEPP and PEPP with time-bin DOF, and the fidelities F' after first-step purification by our scheme and the conventional purification operated once. (b) The fidelity improvement ratio F'/F of the first-step purification. (c) Steps needed to improve the fidelity to 99% by the two schemes. (d) Final efficiencies of the two schemes.

$$
\begin{aligned}
|\Phi_t^{\pm}\rangle &= \frac{1}{\sqrt{2}}\big(e^{i2\phi}|H,t\rangle_A|H,t\rangle_B \pm |V\rangle_A|V\rangle_B\big),\\
|\Psi_t^{\pm}\rangle &= \frac{1}{\sqrt{2}}\big(e^{i2\phi}|H,t\rangle_A|V,t\rangle_B \pm |V\rangle_A|H\rangle_B\big),\\
|\Upsilon_t^{\pm}\rangle &= \frac{1}{\sqrt{2}}\big(e^{i2\phi}|V,t\rangle_A|H,t\rangle_B \pm |H\rangle_A|V\rangle_B\big),\\
|\Gamma_t^{\pm}\rangle &= \frac{1}{\sqrt{2}}\big(e^{i2\phi}|V,t\rangle_A|V,t\rangle_B \pm |H\rangle_A|H\rangle_B\big). \qquad (3)
\end{aligned}
$$

$|\Phi_t^-\rangle$ is got by unilateral σ_z rotation on $|\Phi_t^+\rangle$, which means phase flip. $|\Psi_t^{\mp}\rangle$, $|\Upsilon_t^{\mp}\rangle$ and $|\Gamma_t^{\pm}\rangle$ are got by σ_x rotation on $|\Phi_t^{\pm}\rangle$ on B side, A side, and both A, B sides, which mean bit flips. $\sigma_{x,y,z}$ indicate Pauli rotations (rotations by π rad about the x, y, z axis), which can be used to simulate the perturbations of transmission channels.

After the transmission, we erase the temporal labels by inserting another piece of birefringence material on each side of the pair, with identical length as before, but the axes are reversed from the former one, which means $V(H)$ polarization is along the slow (fast) axis. It compensates the time delay as well as the phase delay, so the eight states in (3) are changed into

$$
\begin{aligned}
|\Phi_t^{\pm}\rangle &\to \frac{1}{\sqrt{2}}\big(|H\rangle_A|H\rangle_B \pm |V\rangle_A|V\rangle_B\big),\\
|\Psi_t^{\pm}\rangle &\to \frac{1}{\sqrt{2}}\big(e^{i\phi}|H\rangle_A|V,t\rangle_B \pm e^{-i\phi}|V\rangle_A|H,-t\rangle_B\big),\\
|\Upsilon_t^{\pm}\rangle &\to \frac{1}{\sqrt{2}}\big(e^{i\phi}|V,t\rangle_A|H\rangle_B \pm e^{-i\phi}|H,-t\rangle_A|V\rangle_B\big),\\
|\Gamma_t^{\pm}\rangle &\to \frac{1}{\sqrt{2}}\big(e^{i2\phi}|V,t\rangle_A|V,t\rangle_B \pm e^{-i2\phi}|H,-t\rangle_A|H,-t\rangle_B\big). \qquad (4)
\end{aligned}
$$

Through suppressing the photons in the earliest and latest time-bins by time window selection, which correspond to the photons in the states of $|-t\rangle$ and $|t\rangle$, only photon pairs in the $|\Phi_t^{\pm}\rangle$ states are left, which are now changed back to be $|\Phi^{\pm}\rangle$. So all the bit flipped noises are wiped off, but the phase flipped noise still exists.

Then, Alice performs a unitary operation

$$
\begin{aligned}
|H\rangle_A &\to \frac{1}{\sqrt{2}}\big(|H\rangle_A - i|V\rangle_A\big),\\
|V\rangle_A &\to \frac{1}{\sqrt{2}}\big(|V\rangle_A - i|H\rangle_A\big) \qquad (5)
\end{aligned}
$$

on each of her qubits, while Bob performs the inverse operation

$$
\begin{aligned}
|H\rangle_B &\to \frac{1}{\sqrt{2}}\big(|H\rangle_B + i|V\rangle_B\big),\\
|V\rangle_B &\to \frac{1}{\sqrt{2}}\big(|V\rangle_B + i|H\rangle_B\big) \qquad (6)
\end{aligned}
$$

of his. $|\Phi^-\rangle$ is changed into $|\Psi^-\rangle$, while $|\Phi^+\rangle$ is unaltered. Then, one can use the conventional iterative purification in Ref. [7] to further distill the mixture, and the state converges to the mixture of $|\Phi^+\rangle$ and $|\Psi^+\rangle$. If the percentage of $|\Phi^+\rangle$ is larger than that of $|\Psi^+\rangle$, the further distillation can rise the percentage of $|\Phi^+\rangle$ to any degree. While if the percentage of $|\Psi^+\rangle$ is larger, one can change $|\Psi^+\rangle$ to $|\Phi^+\rangle$ through proper rotation.

Generally speaking, our first step method deals with a single pair at a time, without the need for target pairs. Hence our scheme is intrinsically very efficient. Moreover, it lowers down the percentages of $|\Psi^{\pm}\rangle$ to zero, which makes the following iterative purification more efficient. Technically, this scheme uses only birefringence material, amplitude modulator to select the right time window, hence is very easy to implement experimentally. It does not require the quantum memory to store the photon pairs. This simplifies the experimental realization drastically.

We show the performance of our purification scheme in realistic situation. Pure PEPPs in state $|\Phi^{+}\rangle$ have to transmit through polarization perturbed channels. To improve the final fidelity to 99%, one scheme is transmitting using $|\Phi^{+}\rangle$ state, followed by the conventional iterative purification, the other is our transmission and distillation methods followed by the iterative purification. The comparisons are made on the iteration steps and the purification efficiency. Two examples are given below, one for unilateral perturbed channels, and the other for bilateral perturbed channels. The unperturbed state and the final desired state are $|\Phi^{+}\rangle$. The purification efficiency is calculated with the method in Ref. [7].

Firstly, we consider the simple case where the noises appear only in one quantum channel, for example only B photon in a pair is affected. After the time delay, unilateral operations of the Pauli matrices, each with the probability of p, on the transmission channel for B photon, change the pure $|\Phi_t^{\pm}\rangle$ states into a Werner state [9] expressed by

$$W = F|\Phi_t^{+}\rangle\langle\Phi_t^{+}| + \frac{1-F}{3}|\Phi_t^{-}\rangle\langle\Phi_t^{-}| + \frac{1-F}{3}|\Psi_t^{+}\rangle\langle\Psi_t^{+}| + \frac{1-F}{3}|\Psi_t^{-}\rangle\langle\Psi_t^{-}|, \tag{7}$$

whose fidelity is $F = 1-3p$ [10] in both cases. For PEPPs, the result is the same, with $|\Phi_t^{\pm}\rangle$ and $|\Psi_t^{\pm}\rangle$ in (7) replaced by $|\Phi^{\pm}\rangle$ and $|\Psi^{\pm}\rangle$. After the first step purification by our method, the mixture is changed into

$$W' = F|\Phi^{+}\rangle\langle\Phi^{+}| + \frac{1-F}{3}|\Phi^{-}\rangle\langle\Phi^{-}|, \tag{8}$$

which could be further distilled except for $F = (1-F)/3$.

Fig. 1(a) shows the fidelities of the perturbed state and those after the first step purifications by the two schemes. The horizontal axis is the random rotation probability p. In our scheme, $F' = F/[F + (1-F)/3]$, while F' got by conventional scheme is given by Eq. (7) in Ref. [5]. Clearly the final fidelity F' of our scheme is larger than the conventional scheme in most cases. Fig. 1(b) denotes the fidelity improvement ratio F'/F logarithmically. The fidelity improvement ratio F'/F is always larger than 1 in our scheme, which means there is no threshold on the initial fidelity for the purification to work, because it cuts down the number of photon pairs in the undesired states without reducing the number of pairs in the desired state. While in the conventional scheme, only mixture with fidelity larger than 50% can be purified. Fig. 1(c) shows steps needed to get a fidelity up to 99% using iterative purification following the first step purification. In most cases our method needs fewer steps than the conventional one dose. Fig. 1(d) is the final purification efficiency. Our scheme also has much higher efficiency. The results also show that the conventional iterative purification is disabled when the perturbation probability is larger than 1/6, while our scheme can work in most cases, except for the singularity at 1/4. The singularity corresponds to the case when the ratios of $|\Phi^{+}\rangle$ and $|\Phi^{-}\rangle$ are equal after the first step purification, which means the fidelity of $|\Phi^{+}\rangle$ at the beginning of conventional iterative purification strategy is 1/2, and the iterative process fails to convergence.

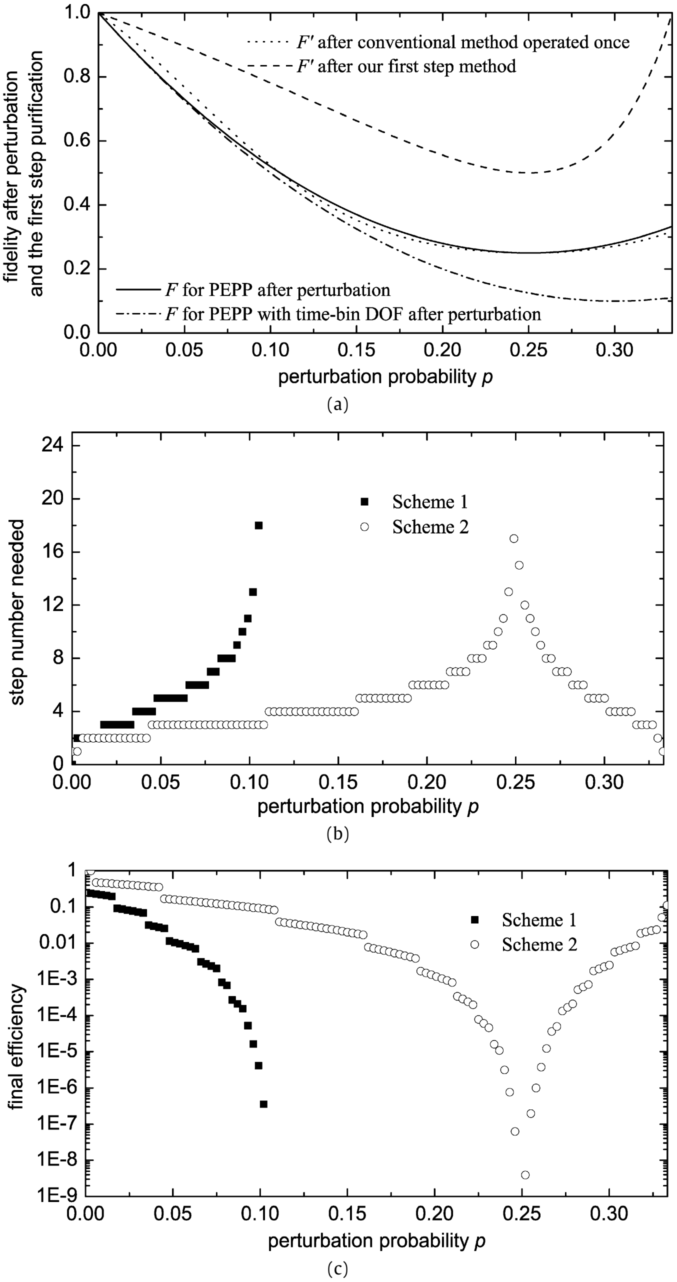

Fig. 2. The comparison between two schemes when the entangled photon pairs are exposed to double sides perturbations. (a) The fidelities F after the states been perturbed, and the fidelities F' after first-step purification by our scheme and the conventional purification operated once. (b) Steps needed to improve the fidelity to 99% by the two schemes. (c) Final efficiencies of the two schemes.

Secondly, we consider a more general case when both photons in the pure entangled pair are exposed to noisy channels. The effects of noises are expressed by the Pauli matrices operating on both A and B sides, each with a probability of p. The corresponding mixture after the same perturbation operating on $|\Phi^{+}\rangle$ state is a Werner state in the same form of (7), but its fidelity becomes $F = 12p^2 - 6p + 1$. While the perturbed state with entanglement on temporal DOF becomes a non-Werner state shown in

$$\begin{aligned} W_d = {} & F|\Phi_t^{+}\rangle\langle\Phi_t^{+}| + R_1|\Phi_t^{-}\rangle\langle\Phi_t^{-}| + R_2|\Psi_t^{+}\rangle\langle\Psi_t^{+}| + R_2|\Psi_t^{-}\rangle\langle\Psi_t^{-}| \\ & + R_2|\Upsilon_t^{+}\rangle\langle\Upsilon_t^{+}| + R_2|\Upsilon_t^{-}\rangle\langle\Upsilon_t^{-}| + R_3|\Gamma_t^{+}\rangle\langle\Gamma_t^{+}| + R_3|\Gamma_t^{-}\rangle\langle\Gamma_t^{-}|, \\ F = {} & 10p^2 - 6p + 1, \\ R_1 = {} & 2p - 6p^2, \qquad R_2 = p - 2p^2, \qquad R_3 = 2p^2. \end{aligned} \tag{9}$$

Fig. 2(a) shows the fidelities F of the perturbed states. Although the fidelity of the state with entanglement on temporal DOF is smaller than that of PEPP after the same perturbation, our purification scheme can greatly improve the fidelity by discarding most of the undesired pairs, and $F' = F/(F + R_1)$. Fig. 2(b), (c) compare the steps needed to reach a fidelity of 99%, and the final efficiency. Our scheme also has great predominance. The threshold for conventional method becomes more critical, but our method is still feasible in most cases. The reason of the singularities in Fig. 2(b), (c) is the same as that in Fig. 1(c), (d).

In conclusion, we have proposed an efficient scheme polarization entanglement purification. Through introducing the entanglement on temporal DOF on the PEPPs, our purification method followed by conventional iterative purification, can improve the fidelity of the mixture to any degree. The above analysis has shown that our scheme has many advantages. The first step purification method deals with a single pair at a time without sacrificing another pair as target, so the efficiency is high. The purification operation only needs modulator and birefringence materials, thus is easy to implement. It has potential applications in most of the quantum information protocols.

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