Efficient strategy for sharing entanglement via noisy channels with doubly entangled photon pairs

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Polarization-entangled photon pairs are easily perturbed in noisy channels. We propose an efficient strategy for sharing polarization-entangled photon pairs (PEPPs) using the additional frequency labels of polarization-frequency doubly entangled photon pairs (DEPPs). The DEPPs are used in transmission, followed by a two-step operation. In the first step, all the bit-flip noises are wiped off efficiently. In the second step, the frequency labels of the DEPPs are erased and the phase-flip noises are wiped off; thus PEPPs in the desired state are extracted. Theoretical analysis shows that this strategy has intrinsically high efficiency, which demonstrates that it has great potential in sharing entanglement via noisy transmission channels.

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\[ |\Phi^+\rangle = \frac{\sqrt{2}}{2}(|H\rangle |H\rangle + |V\rangle |V\rangle), \]

where \(|H\rangle\) and \(|V\rangle\) represent the horizontal and vertical polarizations, respectively. The DEPPs used for transmission are in the state

\[ |\Phi^\prime_D\rangle = \frac{\sqrt{2}}{2}(|H,\omega_i\rangle |H,\omega_i\rangle + |V,\omega'_i\rangle |V,\omega'_i\rangle), \]

where \(s\) and \(i\) represent the signal and idler lights, respectively, according to the usage in nonlinear optics. \(\omega_i\) and \(\omega'_i\) are the possible frequencies of signal photons, while \(\omega_i\) and \(\omega'_i\) are those of idler photons. It can be seen that the state is entangled in both polarization and frequency. When DEPPs in a pure \(|\Phi^\prime_D\rangle\) state are transmitted, the state will be transformed into a mixture composed of the following states:

\[ |\Phi_D\rangle = \frac{1}{\sqrt{2}}(|H,\omega_i\rangle |H,\omega_i\rangle + e^{i\theta}|V,\omega'_i\rangle |V,\omega'_i\rangle), \]
\[ |\Psi_D\rangle = \frac{1}{\sqrt{2}}(|H,\omega_i\rangle |V,\omega_i\rangle + e^{i\theta}|V,\omega'_i\rangle |H,\omega'_i\rangle), \]
\[ |\Gamma_D\rangle = \frac{1}{\sqrt{2}}(|V,\omega_i\rangle |V,\omega_i\rangle + e^{i\theta}|H,\omega'_i\rangle |H,\omega'_i\rangle), \]

in which the \(\theta\)'s are arbitrary phases. \(|\Phi_D\rangle\) includes the phase-flip noise, while \(|\Psi_D\rangle, |\Gamma_D\rangle\) are the bit-flip noises.

The first-step operation is depicted in Fig. 1. There is a polarization-independent wavelength division multiplexer (WDM) on each side of the DEPPs, which guides the photons to the right paths according to their frequencies. Then the photons are injected into a frequency-independent polarization beam splitter (PBS) on each side, which transmits the

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photon pairs in the states of $|\phi^{+}\rangle$ and $|\phi^{-}\rangle$. The desired states are $|\phi^{+}\rangle$ in (3), which are retained in P1 and P2.

Using the frequency DOF of the DEPP as an additional label to select the paths of photons with different polarizations, the first-step operation handles one DEPP at a time, without the need of a target pair; hence it is intrinsically efficient. Moreover, there is no threshold for the initial efficiency.

The DEPPs with phase-flip noise are further distilled and transformed into $|\Phi^{+}\rangle$ in the second step, which is depicted in Fig. 2. Suppose two neighboring pairs, namely, A and B, are exposed to the same phase perturbation [10]; they are both in $|\Phi_{D}\rangle$. First, the state of the pair A is rotated to $|\Gamma_{D}\rangle$ by the half-wave plate (HWP) on both sides. Then, the photons of the two pairs are injected into the PBS on each side from different directions. On the signal side, the photon with the frequency of $\omega_{s}$, whose state is either $|H, \omega_{s}\rangle$ or $|V, \omega_{s}\rangle$, reaches P1, while the photon with the frequency of $\omega_{i}^{'},$ whose state is either $|H, \omega_{i}^{'}\rangle$ or $|V, \omega_{i}^{'}\rangle$, reaches P3. Similarly, on the idler side, the photon in the state either $|H, \omega_{i}\rangle$ or $|V, \omega_{i}\rangle$ reaches P2, while the photon in either $|H, \omega_{i}^{'}\rangle_A$ or $|V, \omega_{i}^{'}\rangle_B$ reaches P4. Since each port is for photons with a certain frequency, in the following analysis, the photon frequencies are replaced by the port numbers for clarity. The whole four-photon state can be expressed as

$$\Pi = |\Phi_{D}\rangle \otimes |\Gamma_{D}\rangle_A = \frac{1}{2} (e^{i\phi}[\Pi_1] + |\Pi_2\rangle + e^{2i\phi}[\Pi_3]),$$

where

$$[\Pi_1] = |H, P1\rangle_B |H, P2\rangle_B |H, P3\rangle_A |H, P4\rangle_A$$

$$+ |V, P1\rangle_A |V, P2\rangle_A |V, P3\rangle_B |V, P4\rangle_B,$$

$$[\Pi_2] = |H, P1\rangle_B |H, P2\rangle_B |V, P3\rangle_A |V, P4\rangle_A,$$

$$[\Pi_3] = |V, P3\rangle_B |V, P4\rangle_B |H, P3\rangle_A |H, P4\rangle_A.$$ (5)

The photons sent out from P1 and P2 are saved, while the photons sent out from P3 and P4 are locally projected onto the diagonal bases, which are expressed as $|D^{\pm}\rangle = (|H\rangle \pm |V\rangle)/\sqrt{2}$, by a HWP and a PBS in each port, and then detected. If the four-photon state is in $[\Pi_3]$, both detectors in P3 and P4 click twice, showing that there is no photon sent out from P1 or P2. Hence only the four-port state $[\Pi_1]$ needs to be considered, in which only one photon is sent out from each port. $[\Pi_1]$ can be further expressed as

$$[\Pi_1] = |\Phi^{+}\rangle_{1,2} \otimes |\Phi^{+}\rangle_{3,4} + |\Phi^{-}\rangle_{1,2} \otimes |\Phi^{-}\rangle_{3,4},$$

where

$FIG. 1.$ Sketch of the first-step distillation for the perturbed DEPPs. The desired states are $|\Phi^{+}\rangle$ in (3), which are retained in P1 and P2.

$FIG. 2.$ Sketch of the second-step operation. The desired state is $|\Phi^{+}\rangle$ in (1), which is sent out from P1 and P2. HWP1 and HWP2 are two half-wave plates that change the state of pair A from $|\Phi_{D}\rangle$ to $|\Gamma_{D}\rangle$. HWP3, PBS3, HWP4, and PBS4 are half-wave plates and polarization beam splitters projecting the state of the photons in P3 and P4 onto the diagonal bases, in order to distinguish $|\Phi^{+}\rangle_{3,4}$ and $|\Phi^{-}\rangle_{3,4}$. HWP5 in P1 is a half-wave plate changing the state of the final PEPPs from $|\Phi^{+}\rangle$ to $|\Phi^{-}\rangle$ if necessary.
two-step operation. In the analysis, we focus on the fidelities 
complementary bases of the doubly entangled states include 
transmission. The state is changed into a mixture by pertur-
This step is 25%, since half pairs are sacrificed as targets and 
is 100% after the second-step operation. The efficiency of 
anterior correlation, i.e., one photon in |D⟩ and the other in |D⟩,
the PEPP in P1 and P2 is in the |Φ⟩ state. On the contrary, if the combined detection result in P3 and P4 is 
PEPPs sent out from P1 and P2, while |Φ⟩ are states of PEPPs sent out from P3 and P4. Equation 
6 shows that, if the result of the combined detection in 
P3 and P4 is correlated, i.e., both photons in |D⟩ or both in 
|D⟩, the PEPP in P1 and P2 is in the |Φ⟩ state. On the contrary, if the combined detection result in P3 and P4 is 
|Φ⟩, which can be converted to |Φ⟩ by a HWP in P1. As a result, the frequency labels are erased and the PEPPs in the desired pure state |Φ⟩ 
are obtained in P1 and P2.

On the assumption that the two neighboring pairs are ex-
posed to the same perturbation, the fidelity of the output state is 
100% after the second-step operation. The efficiency of 
this step is 25%, since half pairs are sacrificed as targets and the probability of the four-photon state in |Π⟩ is 50%. However, the neighboring pairs may not always be in 
the same state. The probability of the neighboring pairs being 
identical can be measured using a fraction of the pairs, which is 
defined as the parameter $g$. The final fidelity of |Φ⟩ equals $g$.
We define the ratio of the dominating state as the initial fidelity of this step operation, which is represented by $F_0$. If 
$g > F_0$, the fidelity is improved since the ratio of |Φ⟩ after 
the operation of this step equals $g$. Otherwise, the effect 
of the second-step operation is only to erase the frequency 
lables. The conventional iterative purification [7] can be used 
for further fidelity improvement. In any case, in a real envi-
ronment, the rate of perturbations due to the environmental variations is always lower than the bit rate. Hence, high 
fidelity could be recovered since $g$ is always close to 1 no 
matter what is the value of $F_0$.

The strategy is feasible for any mixture composed of the states in Eq. (3). To show the performance of the strategy in realistic perturbed channels, two specific examples are ana-
lyzed, in which DEPPs in the pure |Φ⟩ state are used in transmis-
sion. The state is changed into a mixture by pertur-
bations, and then distilled and converted to |Φ⟩ using the 
two-step operation. In the analysis, we focus on the fidelities 
and efficiencies of the operation. The parameter $g$ is taken as 
1, supposing that the bit rate of DEPPs is much faster than the perturbations due to the slow environmental variation. This means that the fidelity of the output state is 100%. The performance of the conventional iterative purification on PEPPs exposed to the same perturbation is also calculated for comparison. The perturbations in the noisy channels can be simulated by Pauli rotations, which are denoted by $σ_y,z$ for rotation of $π$ rad about the $x,y,z$ axes, respectively. The complementary bases of the doubly entangled states include

$$|Φ^+⟩_{D} = \frac{1}{\sqrt{2}}(|H,ω⟩_H|H,ω⟩_V ± |V,ω⟩_H|V,ω⟩_V),$$

$$|Ψ^+⟩_{D} = \frac{1}{\sqrt{2}}(|H,ω⟩_H|V,ω⟩_V ± |V,ω⟩_H|H,ω⟩_V),$$

and $|Φ⟩$, $|Ψ⟩$ are transformed from $|Φ⟩$ by unilateral $σ_z$ 
rotation, while $|Ψ⟩$ is transformed from $|Ψ⟩$ by $σ_x$ rotations on the idler photon, signal photon, and both of them, respectively.

In the first example, the perturbation appears only in a single side of the quantum channel, for example, the idler side. Unilateral perturbations, which are simulated by the three Pauli matrices each with a probability of $p$, change the pure $|Φ⟩$ state into

$$W = F(|Φ⟩⟨Φ|) + \frac{1−F}{3} |Φ⟩⟨Φ|$$

$$+ \frac{1−F}{3} |Ψ⟩⟨Ψ| + \frac{1−F}{3} |Ψ⟩⟨Ψ|,$$

where $F = 1−3p$ is the fidelity of $|Φ⟩$ defined as the ratio of 
DEPPs in $|Φ⟩$. According to the above description, the fidelity of $|Φ⟩$ in the mixture after the first-step operation is $F' = (1+2F)/3$. If the PEPPs in the pure state $|Φ⟩$ are transmitted and exposed to the same unilateral perturbations, it is changed into a mixture in the same form as (9), in which $|Φ⟩$ and $|Ψ⟩$ should be replaced by $|Φ⟩$ and $|Ψ⟩$. The fidelity of $|Φ⟩$ after perturbation is denoted by $F_p$, and the fidelities after the first and second iterations of the purification scheme in Ref. [7] are denoted by $F_p$, and $F_p$, respectively. Figure 3(a) shows the fidelities of $F$, $F'$, $F_p$, and $F_p$, under different unilateral perturbation probabilities. $F_p$ is the same as $F$, while $F_p$, and even $F_p$, are much lower than $F'$ in most cases. The fidelity improvement ratios of the first-step purification in both the DEPP and PEPP cases are plotted in Fig. 3(b). It can be seen that $F'/F$ for our strategy is always larger than 1, which means that there is no threshold for the initial fidelity, while $F_p/F$ for the conventional purification is larger than 1 only if $p > 1/2$.

In our strategy, the first two terms in (9) remain after the first-step operation, with an efficiency of $η = (1+2F)/3$. Then the mixture can be further distilled and converted to $|Φ⟩$ by the second-step operation with an efficiency of 25%. Hence, the total efficiency of our strategy is $η = (1+2F)/12 = (1−2p)/4$. Figure 3(c) shows the efficiencies of $η$ and $η_p$ under different unilateral perturbation probabilities. As a comparison, the efficiencies of the iterative purification scheme for PEPPs after the first and second iterations are also plotted in Fig. 3(c), denoted by $η_p$, and $η_p$, respectively. It can be seen that $η$ is higher than $η_p$, while the total efficiency of our strategy $η$ is higher than the efficiency of the iterative purification of PEPPs with the number of iterations larger than 2.

The more general case is that the DEPPs are exposed to bilateral perturbations, which are simulated by the Pauli ma-
trices operating on both sides, each with a probability of $p$. The mixture after perturbation can be expressed as
The fidelity of \(|\Phi_D^+\rangle\) after perturbation is denoted by \(F\). After the first-step operation of our strategy, the fidelity of \(|\Phi_D^+\rangle\) is improved to \(F' = \frac{F}{1 + R_1}\). In the case of PEPP transmission, the pure \(|\Phi^+\rangle\) state is changed into a mixture in the same form as \(|\Phi^+_p\rangle\), in which \(|\Phi^+_D\rangle\) and \(|\Psi^-\rangle\) should be replaced by \(|\Phi^+_D\rangle\) and \(|\Psi^-\rangle\). The fidelity of \(|\Phi^+_D\rangle\) is \(F_p = 12p^2 - 6p + 1\). The fidelities after the first and second iterations of the purification scheme in Ref. 7 are denoted by \(F_p,1\) and \(F_p,2\), respectively. Figure 4 shows the fidelities \(F, F', F_p, F_p,1, F_p,2\) under different bilateral perturbation probabilities. Although \(F\) for DEPPs is lower than \(F_p\) for PEPPs, after the first-step operation, \(F'\) is much higher than \(F_p,1\) and \(F_p,2\) in most cases.

As in the unilateral perturbation case, only the first two terms in (10) are left after the first-step operation, with an efficiency of \(\eta = F + R_1 = (1 - 2p)^2\). Then the mixture can be further fully distilled and converted to \(|\Phi^+\rangle\) by the second-step operation, with a total efficiency of \(\eta_2 = (1 - 2p)^2/4\). Fig-

\[
W_d = F[|\Phi_D^+\rangle\langle \Phi_D^+| + R_1|\Phi_D^-\rangle\langle \Phi_D^-| + R_2|\Psi_D^+\rangle\langle \Psi_D^+| + R_2|\Psi_D^-\rangle\langle \Psi_D^-| + R_2|\Psi_D^+\rangle\langle \Psi_D^+| + R_2|\Psi_D^-\rangle\langle \Psi_D^-| + R_3|\Gamma_D^+\rangle\langle \Gamma_D^+| + R_3|\Gamma_D^-\rangle\langle \Gamma_D^-|,\]
\]

FIG. 4. Performance comparison between the proposed strategy and the iterative PEPP purification scheme under different bilateral perturbation probabilities. (a) The fidelities versus unilateral perturbation probability. (b) The efficiencies versus unilateral perturbation probability.
ure 4(b) shows the efficiencies of $\eta_1$ and $\eta_2$ under different bilateral perturbation probabilities. As a comparison, the efficiencies of the iterative purification for PEPPs after the first and second iterations are also plotted.

The above analysis is based on the assumption that the whole system is lossless and the detectors are perfect. However, such nonideal conditions would impact the performance of our strategy in practice. According to the feasibility analysis of the conventional PEPP purification schemes [11], the impacts of channel loss and nonideal detection are different in different application environments. In the post-selected environment in which coincidence detection is necessary, the vacuum state and the single photons, as well as the nonideal detection, would not affect the fidelity of the entangled states after our operation. However, the efficiency of the strategy is much lower than expected. In the non-post-selected environment, the effective fidelity should be defined as the ratio of the desired entangled state to the whole output mixture including the entangled states, single-photon states, and the vacuum state, which is highly dependent on the detection efficiency. Our first-step operation may increase the ratio of the single-photon states and the vacuum state to a certain degree, considering the nonideal detection. The impact of limited detection efficiency on the second-step operation, in which the four-port state is reserved according to the detection results, is similar to the analysis in Ref. [10].

In conclusion, we have shown that with DEPPs and the two-step operation, a polarization-entangled state with high fidelity can be achieved after transmission and perturbation. In the first-step operation, the bit-flip noises in the perturbed DEPPs are wiped off using the additional frequency labels. The method is efficient since it handles one pair at a time without the need of target pairs. In the second step, the frequency labels of the output DEPPs of the first-step operation are erased and the PEPPs in the desired state are extracted out, at a cost of half the pairs being sacrificed. The efficiency of the whole process is higher than that of the iterative purification scheme of PEPPs, which means that more qualified pairs can be obtained after transmission and distillation. Hence this strategy has many potential applications in sharing entanglement via noisy transmission channels.

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